Practice Midterm 2-solutions
No homework for the next week. Remember, the Second exam will be on Wednesday, March 25 during the regular class time.

March 22, 2015

1 Problem 1. True or False.

Solution:

1. True.
2. False.
3. False.
4. False.
5. True.

2 Problem 2. A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}$$

(a) Apply the Gram-Schmidt process to find two orthonormal vectors u_1 and u_2 which also span V.

Solution:

By the process of Gram-Schmidt, we just follow the steps:

Firstly, let

$$v_1 = c_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \implies \quad \textit{normalizing} \quad u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and

$$v_2 = c_2 - \frac{c_2^T, v_1}{v_1^T, v_1} v_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \implies \quad \textit{normalizing} \quad u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

(b) Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.

Solution:

The general approach $P = A(A^T A)^{-1} A^T$.

Take

$$Q = (u_1 \quad u_2) = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

since $Q^T Q = I_{2\times 2}$ then $P = QQ^T = u_1 u_1^T + u_2 u_2^T$ is the required projection.

(c) Find the best possible solution to the linear system

$$Q = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Solution:

Minimizing the sum of squared errors $\|Qx - b\|^2$ with respect to x yields

$$Q^T Q x \;=\; Q^T b$$

But $Q^T Q = I_{2\times 2}$ by ortho-normality. So we find

$$x = Q^T b = \begin{pmatrix} 2/\sqrt{3} \\ 5/\sqrt{6} \end{pmatrix}.$$

3 Problem 3. Given

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

(a) Find the matrix P which projects onto the column space of A.

Solution:

Note that the projection on the plane spanned by $(1, 0, 0)^T$ and $(1, 1, 0)^T$ is the same as the projection on the plane spanned by $(1, 0, 0)^T$ and $(0, 1, 0)^T$, so it is simply

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can also use the general approach $P = A(A^TA)^{-1}A^T$, which is longer but gives the same answer.

(b) Compute the projection p of b onto this column space.

Solution:

$$p = Pb = (2,3,0)^T$$

(c) Find the error $e = b - p$ and show that it lies in the left nullspace of A.

Solution:

Apparently $e = b - p = (0,0,4)^T$, and

$$A^Te = \mathbf{0}$$

which means e is in the left null space of A.

4 Problem 4. Calculate the determinant of the matrix A.

Solution:

We calculate step by step:

$$\det A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 6 & 4 & 8 \\ 3 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 2 & -2 & 0 \\ 0 & -5 & -8 & -10 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ 0 & -4 & -8 & -12 \\ 0 & -5 & -8 & -10 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -12 & -12 \\ 0 & 0 & -13 & -10 \end{vmatrix} = -2 \begin{vmatrix} -12 & -12 \\ -13 & -10 \end{vmatrix} = -2(120 - 156) = 72$$

5 Problem 5. If you know all 16 cofactors of a 4 by 4 invertible matrix A, how would you find its determinant $\det(A)$?

Solution:

Since $det(A) \neq 0$, by Cramer's Rule, we have (the Cofactor formula):

$$A^{-1} = \frac{C^T}{\det(A)}.$$

Multiply both sides by A:

$$AA^{-1} = \frac{AC^T}{det(A)} \implies I = \frac{AC^T}{det(A)} \implies det(A)I = AC^T$$

Take the determinant in the last equality:

$$det(A)^4 = det(A)det(C^T) \implies det(A)^3 = det(C^T).$$

Finally,

$$det(A) = det(C^T)^{1/3}.$$